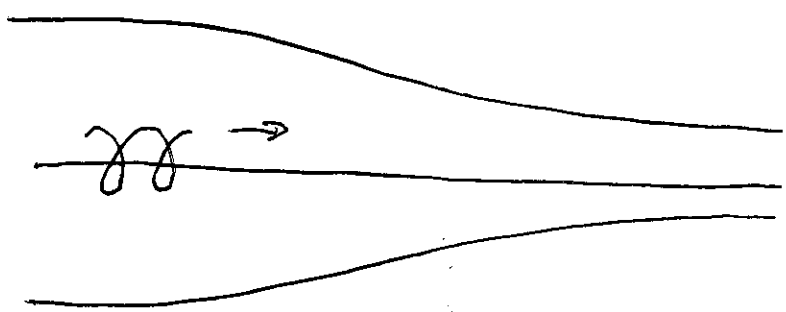
Particle spirals along field which is getting stronger or weaker



If field is static, + no $\vec{E}$ then energy is conserved

$$\frac{dU}{dt} = q\,\vec{E}\cdot\vec{v} = 0 \quad \Rightarrow \quad U = \text{const}$$

$\Rightarrow \gamma = \text{const} \Rightarrow$

i.e. $$\frac{v_\perp^2}{B(z)} = \frac{v_{\perp 0}^2}{B_0}$$

and we have $$v_0^2 = v_\parallel^2 + v_{\perp 0}^2 \frac{B(z)}{B_0}$$

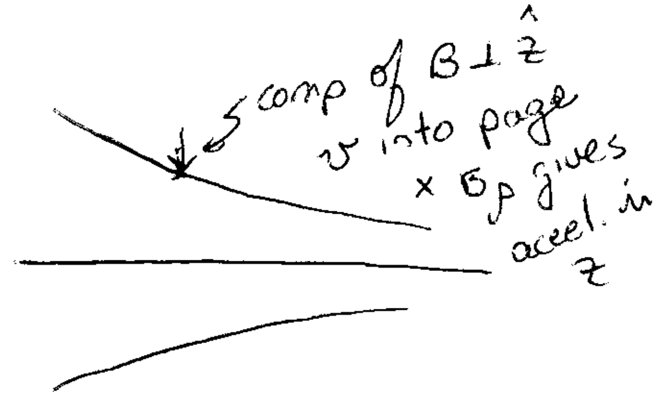
v_0^2 defined as $v^2 = \text{const}$

constants defined so that $v_\parallel = 0$ for $z = z_0$, $B(z_0) = B_0$

Problem reduces to that of motion of single particle in 1D potential $\propto B(z)$

z_0 is a turning point

In terms of Lorentz force



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0th order $\rho = a$ + first equation gives

$$-\rho\dot{\phi}^2 = \frac{q}{\gamma mc}\rho\dot{\phi}B_z \quad \text{or} \quad \dot{\phi} = -\frac{qB}{\gamma mc} \checkmark$$

as well, second gives $\rho^2\dot{\phi} = $ const.

positive ω_B implies neg. $\dot{\phi}$

$$= -a^2\omega_{B,0} = -\frac{v_{\perp 0}^2}{\omega_{B,0}}$$

$$\omega_{B,0} = \frac{qB}{\gamma mc}$$

At 0th or $\ddot{z} = 0$ but small B_ρ cross v_ϕ

implies z force.

$$\ddot{z} = \frac{q}{2\gamma mc}\rho^2\dot{\phi}\frac{dB}{dz}$$

$$\simeq -\frac{v_{\perp 0}^2}{2}\frac{q}{\gamma mc}\frac{1}{\omega_{B,0}}\frac{dB}{dz} = -\frac{v_{\perp 0}^2}{2B_0}\frac{dB}{dz}$$

so $$\frac{1}{2}\dot{z}^2 + \frac{v_{\perp 0}^2}{2}\frac{B(z)}{B_0} = \frac{v_0^2}{2}$$ integration const.

Lagrangian mechanics

non relativistic formalism

$$L = L(q_i, \dot{q}_i, t)$$

$$A = \int_{t_1}^{t_2} L \, dt$$

A an extremum $\quad \delta A = 0$

$$\delta A = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt$$

$$\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \delta q_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

so $$\delta A = \int_{t_1}^{t_2} \left(\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \cancelto{0}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)} \right) dt$$

NR case $L = K - V = \frac{1}{2}mv^2 - V(\vec{x})$

gives $-\vec{\nabla}V - \frac{d}{dt}m\vec{v} = 0$ or $\vec{F} = m\vec{a}$

Now A should be a Lorentz invariant scalar

$$A = \int_{t_1}^{t_2} L\,dt = \int_{\tau_1}^{\tau_2} L \frac{dt}{d\tau}\,d\tau \qquad \tau \text{ proper time}$$

$$= \int_{\tau_1}^{\tau_2} \gamma L\,d\tau$$

so γL must be a Lorentz invariant scalar

for a free particle, only 4-vector is u^α, only scalar is $u^\alpha u_\alpha = -c^2$

$$L_{free} = -\frac{mc^2}{\gamma} = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\simeq -mc^2 + \frac{1}{2}mv^2 \quad \text{NR limit}$$

$$\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\frac{mv_i}{(1-v^2/c^2)^{1/2}}\right) = \frac{d}{dt}\gamma m v_i = 0$$

i.e. $p_i = \text{const.}$

Now we include interactions

NR Lagrangian $-$ $K - V$ $\qquad V = e\Phi$

i.e. $L_{int} = -e\Phi$

Less obvious what to do with Lorentz force.

Use relativity. For particle at rest $u^\alpha = (c, 0)$; $A^\alpha = (\Phi, \vec{A})$
+ covariance

so $L_{int} = -\dfrac{e}{\gamma c} u^\alpha A_\alpha$ $\qquad$ then γL_{int} is Lorentz invariant scalar + $L_{int} \simeq -e\Phi$ for particle initially at rest.

$$= -e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A}$$

so the Lagrangian for the Lorentz force is $\dfrac{e}{c}\vec{v}\cdot\vec{A}$

$$L = -mc^2\left(1 - v^2/c^2\right)^{1/2} + \frac{e}{c}\vec{v}\cdot\vec{A} - e\Phi$$

variation gives $\dfrac{d}{dt}\gamma m\vec{v} = e\left(\vec{E} + \dfrac{\vec{v}}{c}\times\vec{B}\right)$

where $\vec{E} = -\vec{\nabla}\Phi - \dfrac{\partial\vec{A}}{\partial t}$ $\quad \vec{B} = \vec{\nabla}\times\vec{A}$ and $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dfrac{d\vec{x}}{dt}\dfrac{\partial}{\partial\vec{x}}$

$$= \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

Canonical momentum $\vec{P}$ conjugate to position $\vec{x}$

$$P_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + \frac{e}{c} A_i = p_i + \frac{e}{c} A_i$$

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$$H = \left(\left(c\vec{P} - e\vec{A} \right)^2 + m^2 c^4 \right)^{1/2} + e\Phi$$

Hamiltonian.

Derive Lorentz force from Hamilton's equations

Covariant treatment

Twin paradox - free particle follows path of maximal proper time

$$A \propto \int_{\tau_1}^{\tau_2} d\tau$$

$$c\,d\tau = \left(\eta_{\alpha\beta}\, dx^\alpha dx^\beta\right)^{1/2} = \left(\eta_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2} ds$$

s is any monotonic parameter along path

$$A = -mc \int_{\tau_1}^{\tau_2} \left(\eta_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2} ds$$

units of ang. momentum same as $\hbar$

$$L_{free} = -mc\left(\eta_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)^{1/2}$$

Euler-Lagrange $\quad \dfrac{d}{ds}\dfrac{\partial L}{\partial \frac{dx^\alpha}{ds}} = \dfrac{\partial L}{\partial x^\alpha}$

for free particle $\frac{\partial L}{\partial x^\alpha} = 0$ and we have

$$\frac{\partial L}{\partial x^\alpha} = -\frac{e}{c}\frac{dx^\beta}{ds}\frac{\partial A_\beta}{\partial x^\alpha}$$

$$\frac{\partial L}{\partial \frac{dx^\alpha}{ds}} = -mc\,\frac{\frac{dx_\alpha}{ds}}{\left(\eta_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right)^{1/2}} - \frac{e}{c}A_\alpha(x)$$

$$= -m\frac{dx_\alpha}{d\tau} - \frac{e}{c}A_\alpha(x)$$

Adiabatic invariants

q_i generalize coordinate

p_i conjugate momentum

$$\oint p_i dq_i \simeq \text{constant} = J$$

e.g. pendulum with changing length

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\simeq \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\frac{\theta^2}{2} \qquad \theta = \theta_0 \cos\omega t \quad \omega = \left(\frac{g}{l}\right)^{1/2}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$J_\theta = \oint p_\theta dq_\theta = \int_0^T m l^2 \dot{\theta} d\theta \propto m l^2 \omega \theta_0^2 = m l^{3/2} g^{1/2} \theta_0^2$$

So $\quad l^{3/2}\theta_0^2 = \text{const}$

$$E = ml^2\dot{\theta}^2 = ml^2\theta_0^2\omega^2$$ so E/ω is approx const.